An Algorithm for the Derivation of Rapidly Converging Infinite Series for Universal Mathematical Constants

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Abstract

In this paper I'll derive a novel algorithm that can be used to derive rapidly converging infinite series for various mathematical constants such as pi and values of the gamma function. A compendium of new infinite series is given.

Introduction

The purpose of this paper is lay out a novel, explicit algorithm that can be used to derive elegant, rapidly converging infinite series for various mathematical constants. New formulas for pi, gamma function, and other constants are derived inspired by David & Peter Borwein [2], Simon Plouffe, Travis Sherman [3], Christian Krattenthaler [4], and Boris Gour´evitch & Jes´us Guillera Goyanes [1], Bellard [6] who wrote similar papers based on deriving infinite series for constants.

Examples of formulas that can be derived though the algorithm include:

$$\pi = \frac{\sqrt{3}}{60}\sum_{n=0}^{\infty}\frac{(2n)!(130n+109)}{\left(\frac{7}{6}\right)_n\left(\frac{11}{6}\right)_n(-1296)^n} \tag{1.1}$$

And

$$\frac{7\cdot 3^{1/2}\left(\Gamma(1/3)\right)^3}{\pi\cdot 2^{1/3}} = \sum_{n=0}^{\infty}\frac{\left(\frac{2}{3}\right)_n\left(\frac{1}{2}\right)_n(102n+59)}{\left(\frac{13}{12}\right)_n\left(\frac{19}{12}\right)_n(-288)^n} \tag{1.2}$$

These formulas were derived with the following integrals which we shall prove in this paper:

$$\frac{z\cdot\Gamma\left(a+b+2\right)}{\Gamma\left(a+1\right)\Gamma\left(b+1\right)}\int_0^1\frac{x^a\left(1-x\right)^b}{z-x^k\left(1-x\right)^s}dx=\sum_{n=0}^{\infty}\frac{\left(a+1\right)_{kn}\left(b+1\right)_{sn}}{\left(a+b+2\right)_{\left(k+s\right)n}\left(z\right)^n} \tag{1.3}$$

We'll also prove the more comprehensive integral-polynomial relation. This is the so called 'machinery' behind these infinite series:

$$\frac{\Gamma\left(a+1\right)\Gamma\left(b+1\right)}{z\cdot\Gamma\left(a+b+2\right)}\sum_{n=0}^{\infty}\left(\frac{\left(a+1\right)_{kn}\left(b+1\right)_{sn}}{\left(a+b+2\right)_{\left(k+s\right)n}\left(z\right)^n}\left(w\right)\right)=$$

$$\int_0^1\frac{Q\left(x\right)x^a\left(1-x\right)^b}{z-x^k\left(1-x\right)^s}dx=\int_0^1\frac{x^a\left(1-x\right)^b}{P\left(x\right)}dx=c \tag{1.4}$$

$$w=\left\{\begin{array}{l} a_p\prod_{g=1}^{p}\left(\frac{a+g+kn}{a+b+g+1+\left(k+s\right)n}\right)+a_{p-1}\prod_{g=1}^{p-1}\left(\frac{a+g+kn}{a+b+g+1+\left(k+s\right)n}\right)+ \\ ...+a_1\left(\frac{a+1+kn}{a+b+2+\left(k+s\right)n}\right)+a_0 \end{array}\right\}$$

Derivation of Algorithm

We will begin by deriving formula (1.3) and apply it to derive the more complicated identity (1.4)

Following the steps outlined in Boris Gour´evitch, Jes´us Guillera Goyanes' paper ref [1], first consider the beta function:

$$\int_0^1 x^{kn+a}\left(1-x\right)^{sn+b}dx=\frac{\Gamma\left(kn+a+1\right)\Gamma\left(sn+b+1\right)}{\Gamma\left(\left(k+s\right)n+a+b+2\right)}$$

Then divide both sides by z^n and take the summation

$$\sum_{n=0}^{\infty}\left(\int_0^1 \frac{x^{kn+a}(1-x)^{sn+b}}{z^n}dx\right)=\sum_{n=0}^{\infty}\frac{\Gamma(kn+a+1)\Gamma(sn+b+1)}{\Gamma(a+b+2+n(s+k))\cdot z^n}$$

This is equal to:

$$\sum_{n=0}^{\infty}\left(\int_0^1 x^a(1-x)^b\left(\frac{x^k(1-x)^s}{z}\right)^n dx\right)=\sum_{n=0}^{\infty}\frac{\Gamma(kn+a+1)\Gamma(sn+b+1)}{\Gamma(a+b+2+n(s+k))\cdot z^n} \qquad (2.0)$$

Summing an infinite geometric series as shown in ref [1]:

$$\sum_{n=0}^{\infty}\left(\int_0^1 x^a(1-x)^b\left(\frac{x^k(1-x)^s}{z}\right)^n dx\right)=z\int_0^1 \frac{x^a(1-x)^b}{z-x^k(1-x)^s}dx \qquad (2.1)$$

Using the identity: $(\Gamma(w))(w)_n=\Gamma(w+n)$ we derive:

$$\frac{\Gamma(kn+a+1)\Gamma(sn+b+1)}{\Gamma((k+s)n+a+b+2)}=\frac{(a+1)_{kn}(b+1)_{sn}\Gamma(a+1)\Gamma(b+1)}{(a+b+2)_{(k+s)n}\Gamma(a+b+2)} \qquad (2.2)$$

Plugging equations (2.1) and (2.2) into (2.0) and re-arranging terms we get equation (1.3)

To make the integral useful for computing constants begin with a so called ‘seed integral’. Such an integral must be in the form:

$\int_0^1 \frac{x^a(1-x)^b}{P(x)}dx=c$ where $P(x)$ is a polynomial and c is a constant

Then choose integer values of k and s such that the polynomial expansion of $z-x^k(1-x)^s$ is divisible by $P(x)$. The value of z is determined by computing the quotient polynomial $Q(x)$ via polynomial long division such that the following identity is obtained:

$$\int_0^1 \frac{x^a(1-x)^b}{P(x)}dx=\int_0^1 \frac{Q(x)x^a(1-x)^b}{z-x^k(1-x)^s}dx$$

$Q(x)$ can be generalized to: $Q(x)=a_px^p+a_{p-1}x^{p-1}...a_1x+a_0$

Multiplying $\int_0^1 \frac{x^a(1-x)^b}{z-x^k(1-x)^s}dx$ by $Q(x)$ and splitting the a_p coefficients of $Q(x)$ yields:

$$\left\{\begin{array}{l} a_p\int_0^1 \frac{x^{a+p}(1-x)^b}{z-x^k(1-x)^s}dx + a_{p-1}\int_0^1 \frac{x^{a+p-1}(1-x)^b}{z-x^k(1-x)^s}dx \\ ....a_1\int_0^1 \frac{x^{a+1}(1-x)^b}{z-x^k(1-x)^s}dx + a_0\int_0^1 \frac{x^a(1-x)^b}{z-x^k(1-x)^s}dx \end{array}\right\} = \int_0^1 \frac{x^a(1-x)^b}{P(x)}dx$$

Each of these $a_p \ldots a_0$ split terms can be plugged into equation (1.3) :

$$\left\{\begin{array}{l} a_p\int_0^1 \frac{x^{a+p}(1-x)^b}{z-x^k(1-x)^s}dx = \frac{a_p\cdot\Gamma(a+p+1)\Gamma(b+1)}{z\cdot\Gamma(a+b+p+2)}\sum_{n=0}^{\infty}\frac{(a+p+1)_{kn}(b+1)_{sn}}{(a+b+p+2)_{(k+s)n}(z)^n} \\ a_{p-1}\int_0^1 \frac{x^{a+p-1}(1-x)^b}{z-x^k(1-x)^s}dx = \frac{a_{p-1}\cdot\Gamma(a+p)\Gamma(b+1)}{z\cdot\Gamma(a+b+p+1)}\sum_{n=0}^{\infty}\frac{(a+p)_{kn}(b+1)_{sn}}{(a+b+p+1)_{(k+s)n}(z)^n} \\ \bullet \\ \bullet \\ \bullet \\ a_1\int_0^1 \frac{x^{a+1}(1-x)^b}{z-x^k(1-x)^s}dx = \frac{a_1\cdot\Gamma(a+2)\Gamma(b+1)}{z\cdot\Gamma(a+b+3)}\sum_{n=0}^{\infty}\frac{(a+2)_{kn}(b+1)_{sn}}{(a+b+3)_{(k+s)n}(z)^n} \\ a_0\int_0^1 \frac{x^{a}(1-x)^b}{z-x^k(1-x)^s}dx = \frac{a_0\cdot\Gamma(a+1)\Gamma(b+1)}{z\cdot\Gamma(a+b+2)}\sum_{n=0}^{\infty}\frac{(a+1)_{kn}(b+1)_{sn}}{(a+b+2)_{(k+s)n}(z)^n} \end{array}\right\} \quad (2.3)$$

Also note the following pochhammer and gamma function identities:

$$(a+b+p+2)_{(k+s)n}=\left(\prod_{g=1}^{p}\left(\frac{a+b+g+1+(k+s)n}{a+b+g+1}\right)\right)(a+b+2)_{(k+s)n}$$

$$(a+p+1)_{kn}=\left(\prod_{g=1}^{p}\left(\frac{a+g+kn}{a+g}\right)\right)(a+1)_{(kn)}$$

$$\Gamma(a+b+p+2)=\left(\prod_{g=1}^{p}(a+b+g+1)\right)\Gamma(a+b+2) \tag{2.4}$$

$$\Gamma(a+p+1)=\left(\prod_{g=1}^{p}(a+g)\right)\Gamma(a+1)$$

Putting it all together utilizing (2.3) and (2.4) gives:

$$\frac{a_p\cdot\Gamma(a+p+1)\Gamma(b+1)}{z\cdot\Gamma(a+b+p+2)}\sum_{n=0}^{\infty}\frac{(a+p+1)_{kn}(b+1)_{sn}}{(a+b+p+2)_{(k+s)n}(z)^n}=$$

$$\frac{a_p\left(\prod_{g=1}^{p}(a+g)\right)\Gamma(a+1)\Gamma(b+1)}{z\cdot\left(\prod_{g=1}^{p}(a+b+g+1)\right)\Gamma(a+b+2)}\sum_{n=0}^{\infty}\left(\frac{\left(\prod_{g=1}^{p}\left(\frac{a+g+kn}{a+g}\right)\right)(a+1)_{kn}(b+1)_{sn}}{\left(\prod_{g=1}^{p}\left(\frac{a+b+g+1+(k+s)n}{a+b+g+1}\right)\right)(a+b+2)_{(k+s)n}(z)^n}\right)$$

Also note how the $\prod_{g=1}^{p}(a+b+g+1)$ and $\prod_{g=1}^{p}(a+g)$ are eliminated such that we simplify and obtain:

$$\frac{a_p\Gamma(a+1)\Gamma(b+1)}{z\cdot\Gamma(a+b+2)}\sum_{n=0}^{\infty}\left(\frac{\left(\prod_{g=1}^{p}(a+g+kn)\right)(a+1)_{kn}(b+1)_{sn}}{\left(\prod_{g=1}^{p}(a+b+g+1+(k+s)n)\right)(a+b+2)_{(k+s)n}(z)^n}\right)=$$

$$a_p\int_0^1\frac{x^{a+p}(1-x)^b}{z-x^k(1-x)^s}dx$$

Performing this procedure for $a_p\ldots a_0$, summing the p+1 terms, and factoring out $\dfrac{(a+1)_{kn}(b+1)_{sn}}{(a+b+2)_{(k+s)n}(z)^n}$ gives equation (1.4), which completes the proof.

One final pochhammer identity that will be encountered frequently in this paper is

$$(a)_{nk}=\left(\prod_{y=0}^{k-1}\left(\frac{a+y}{k}\right)_n\right)k^{kn} \tag{2.5}$$

Another forumula that will be used extensively later is the Euler integral:

$$2F1\{x_1,x_2;y_1;z\}=\frac{\Gamma(y_1)}{\Gamma(x_2)\Gamma(y_1-x_2)}\int_0^1\frac{x^{x_2-1}(1-x)^{y_1-x_2-1}}{(1-zx)^{x_1}}dx \tag{2.6}$$

Deriving infinite series for a constant (c) is as simple as plugging in values of k, s, a, b, z and the a_p coefficients of $Q(x)$ into formula (1.4).

Deriving Formulas for Pi

We shall use algorithm (1.4) to find various rapidly converging formulas for pi. We'll begin by deriving formula (1.1)

Consider a hypergeometric function for arcsine converted into integral form via (2.6)

$$2F1\left\{1,\frac{1}{2};\frac{3}{2};\frac{w^2}{w^2-1}\right\}=\left(\frac{\sin^{-1}(w)}{w}\right)\sqrt{(1-w^2)}=\frac{1}{2}\int_0^1\frac{x^{-1/2}}{1-\left(\frac{x\cdot w^2}{w^2-1}\right)}dx \tag{2.7}$$

Then let w=1/2 so that $\frac{\pi\sqrt{3}}{3}=\int_0^1 \frac{x^{-1/2}}{1+\frac{x}{3}}dx$

To derive a rapidly converging infinite series for this integral using (1.4) let k=1 and s=2. Then let a=-1/2 b=0 and $P(x)=1+\frac{x}{3}$ and find Q(x) such that:

$$\int_0^1 \frac{x^{-1/2}Q(x)}{z-x(1-x)^2}dx=\int_0^1 \frac{x^{-1/2}}{1+\frac{x}{3}}dx$$

Expand $z-x(1-x)^2$ and perform polynomial division $\frac{x}{3}+1\overline{)-x^3+2x^2-x+z}$

A value of z = -48 and $Q(x)=-3x^2+15x-48$ is obtained. Thus:

$$\int_0^1 \frac{x^{-1/2}\left(x^2-5x+16\right)}{z-x(1-x)^2}dx=\frac{-1}{3}\int_0^1 \frac{x^{-1/2}}{1+\frac{x}{3}}dx=\frac{-\pi\sqrt{3}}{9} \tag{2.8}$$

Then plug the following values into (1.4):

a=-1/2, b=0, k=1, s=2, z=-48, $a_0=16$, $a_1=-5$, $a_2=1$ and obtain:

$$\frac{-\pi\sqrt{3}}{9}=\frac{\Gamma(1/2)\Gamma(1)}{-48\Gamma(3/2)}\sum_{n=0}^{\infty}\frac{(1/2)_n(1)_{2n}}{(3/2)_{3n}(-48)^n}\left(16-5\left(\frac{1/2+n}{3/2+3n}\right)+\left(\frac{1/2+n}{3/2+3n}\right)\left(\frac{3/2+n}{5/2+3n}\right)\right)$$

Using identity (2.5) and simplification on the equation above, formula (1.1) is derived:

$$\pi=\frac{\sqrt{3}}{60}\sum_{n=0}^{\infty}\frac{(2n)!(130n+109)}{\left(\frac{7}{6}\right)_n\left(\frac{11}{6}\right)_n(-1296)^n}$$

Interestingly, formula (1.1) converges at a rate of log(324) or 2.5 digits per term using an arcsine formula for ½, whereas the original arcsine formulas utilizing the binomial theorem converge at best at a rate of only log(4) for ½. Yet formula (1.1) is compact, unique, and converges about four times faster.

A faster formula can be obtained that adds five digits per term. Let s=4, k=2, a=-1/2, and b=0. $P(x)=\left(1+\frac{x}{3}\right)$ Then expand $z-x^2(1-x)^4$ and performing division for Q(x) the following identity is obtained:

$$\int_0^1 \frac{x^{-1/2}\left(-x^5+7x^4-27x^3+85x^2-256x+768\right)}{2304-x^2(1-x)^4}dx=\frac{1}{3}\int_0^1 \frac{x^{-1/2}}{1+\frac{x}{3}}dx=\frac{\pi\sqrt{3}}{9} \qquad (2.9)$$

After some labor a pi formula derived using (1.4), in 'BBP' notation [2]:

$$\pi=\frac{\sqrt{3}}{6^5}\sum_{n=0}^{\infty}\left(\frac{\left[(4n)!\right]^2(6n)!}{(2n)!(12n)!(9)^{n+1}}\right)\left(\frac{127169}{12n+1}-\frac{1070}{12n+5}-\frac{131}{12n+7}+\frac{2}{12n+11}\right) \qquad (2.10)$$

Or using identity (2.5) we can express (2.10) in pochhammer notation:

$$\pi=\frac{\sqrt{3}}{1155}\sum_{n=0}^{\infty}\frac{(4n)!\left(671840n^3+1289936n^2+782458n+150835\right)}{(72)^{4n+1}(13/12)_n(17/12)_n(19/12)_n(23/12)_n} \qquad (2.11)$$

To derive another pi formula let s=2, k=2, a=-1/2, b=0 and $P(x)=1+\frac{x}{3}$ to obtain:

$$\int_0^1 \frac{x^{-1/2}\left(-x^3+5x-16x+48\right)}{144-x^2(1-x)^2}dx=\frac{\pi\sqrt{3}}{9} \qquad (2.12)$$

After some labor using (1.4) we get the BBP form equation below which ads 3.4 digits/term:

$$\pi 2^{10}\sqrt{3}=\sum_{n=0}^{\infty}\frac{1}{\binom{8n}{4n}9^n}\left(\frac{5717}{8n+1}-\frac{413}{8n+3}-\frac{45}{8n+5}+\frac{5}{8n+7}\right) \qquad (2.13)$$

Using Multivariable Values of P(x) and Q(x) to Derive Infinite Series for Pi

A slew of binomial identities for pi and other constants can be found using equations (1.3) and (1.4), but extending it to multiple variables. In this section Q(x) becomes Q(x,w), P(x) becomes P(x,w), and z becomes a function of w.

To begin, using equation (1.3) let a=0, b=0, k=1, s=1, z=w to obtain the identity:

$$\int_0^1 \frac{1}{w-x(1-x)}dx = \sum_{n=0}^{\infty} \frac{1}{w^{n+1}\binom{2n}{n}(2n+1)} \tag{3.1}$$

By evaluating the left hand integral we see that (3.1) is equal to [3]:

$$\frac{4\tan^{-1}\left(1/\sqrt{4w-1}\right)}{\sqrt{4w-1}} \tag{3.2}$$

Do derive a faster series for (3.1) let k=3, s=3, a=0,b=0, z=z, and expand $z-x^3(1-x)^3$
And let P(x,w) = x^2-x+w. Performing polynomial division, we need to find Q(x,w) and a new value of z in terms of w. $\left(x^2-x+w\right)\overline{\left)\left(x^6-3x^5+3x^4-x^3+z\right)\right.}$

We find: $Q(x,w)=\left(x^4-2x^3+(1-w)x^2+xw+w^2\right), z=w^3$ Therefore:

$$\int_0^1 \frac{\left(x^4-2x^3+(1-w)x^2+xw+w^2\right)}{w^3-x^3(1-x)^3}dx = \int_0^1 \frac{1}{w-x(1-x)}dx \tag{3.3}$$

With regards to equation (3.1) we can let $w=w^{-2}$,divide by w, and take the derivative in terms of w such that:

$$\sum_{n=0}^{\infty} \frac{w^{2n+1}}{\binom{2n}{n}(2n+1)}\frac{d}{dw} = \sum_{n=0}^{\infty}\frac{w^{2n}}{\binom{2n}{n}} = \frac{w^2}{4-w^2} + \frac{4w\tan^{-1}\left(w/\sqrt{4-w^2}\right)}{\left(4-w^2\right)^{3/2}} \tag{3.4}$$

An arbitrary number of integrals and derivatives of (3.4) can be taken in terms of w to obtain various central binomial identities.

Applying formula (1.4) on (3.3), and after some labor obtain:
(The variable r denotes the powers of n that results from taking multiple derivatives)

$$4\sum_{n=1}^{\infty}\frac{w^n}{\binom{2n}{n}n^r} =$$

$$2w^2+\frac{2w^2}{3\cdot 2^r}+\sum_{n=1}^{\infty}\frac{w^{3n}}{\binom{6n}{3n}}\left(\frac{w^2\left(9n^2+9n+2\right)}{(3n+2)^r(6n+3)(6n+1)}+\frac{2w(3n+1)}{(6n+1)(3n+1)^r}+\frac{4}{(3n)^r}\right) \tag{3.5}$$

To derive an elegant pi formula let r=0 and $w=w^2$ in (3.5). Then divide (3.5) by w and take multiple derivatives such that the denominators vanish, ensuring you divide (3.5) by w each time before the derivative is taken. Then finally let w=1 to obtain the identity:

$$3\sum_{n=1}^{\infty}\frac{63n^2-27n+4}{\binom{6n}{3n}}=16\sum_{n=1}^{\infty}\frac{n^2}{\binom{2n}{n}}-32\sum_{n=1}^{\infty}\frac{n}{\binom{2n}{n}}+12\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}} \tag{3.6}$$

Using (3.4) and taking multiple derivatives we arrive at:

$$3\sum_{n=1}^{\infty}\frac{63n^2-27n+4}{\binom{6n}{3n}}=\frac{40\pi\sqrt{3}}{81}+4 \tag{3.7}$$

A faster converting series can be obtained by letting k=5, s=5 such that

$$\int_0^1\frac{Q(x,w)}{w^5-x^5(1-x)^5}dx=\int_0^1\frac{1}{P(x,w)}dx$$

$$P(x,w)=w-x(1-x) \tag{3.8}$$

$$Q(x,w)=\begin{pmatrix} x^8-4x^7+(6-w)x^6+(3w-4)x^5+\left(w^2-3w+1\right)x^4 \\ +\left(w-2w^2\right)x^3+\left(w^2-w^3\right)x^2+w^3x+w^4 \end{pmatrix}$$

Using formula (1.4) and after some labor the identity is obtained:

$$\sum_{n=0}^{\infty}\frac{213125n^4-278000n^3+139975n^2-26800n+1596}{\binom{10n}{5n}}=$$

$$16\sum_{n=0}^{\infty}\frac{16n^4-128n^3+344n^2-352n+105}{\binom{2n}{n}}=\frac{1120\pi\sqrt{3}}{81}+1728 \tag{3.9}$$

Rapidly Converging Infinite Series for the Gamma Function

Elegant, rapidly converging infinite series for the gamma function such as (1.2) can be derived using identities (1.3) and (1.4). In this section we'll also prove an 'accidental' quadratic hypergeometric transformation that is a byproduct of this method, as well as other identities.

Begin with Kummer's formula [5] and express it in integral form via (2.6):

$$2F1\{1,h;2-h;-1\}=\frac{\Gamma(2-h)\Gamma(3/2)}{\Gamma(3/2-h)\Gamma(2)}=\frac{\Gamma(2-h)}{\Gamma(2-2h)\Gamma(h)}\int_0^1\frac{x^{h-1}(1-x)^{1-2h}}{1+x}dx \quad (4.1)$$

Since $-2-x(1-x)=(x-2)(x+1)$ we can let k =1, s =1, z = -2 and $Q(x)=x-2$
And after isolating the integral in (4.1) we have the identity:

$$\int_0^1\frac{x^{h-1}(1-x)^{1-2h}(x-2)}{-2-x(1-x)}dx=\int_0^1\frac{x^{h-1}(1-x)^{1-2h}}{x+1}dx=\frac{\Gamma(2-2h)\Gamma(h)\sqrt{\pi}}{2\Gamma(3/2-h)} \quad (4.2)$$

The leftmost integral in (4.3) can be evaluated using (1.4). setting $a_0=-2$ $a_1=1$
a=h-1 b=1-2h :

$$\frac{\Gamma(h)\Gamma(2-2h)}{-2\Gamma(2-h)}\sum_{n=0}^{\infty}\frac{(h)_n(2-2h)_n}{(2-h)_{2n}(-2)^n}\left(\frac{h+n}{2-h+2n}-2\right)=\frac{\sqrt{\pi}\Gamma(h)\Gamma(2-2h)}{2\Gamma(3/2-h)}$$

After some labor and using (2.5) we obtain this quickly converging formula for computing values of the gamma function:

$$\frac{h(h+1)2^{2h-1}\left[\Gamma(h)\right]^2}{\Gamma(2h)}=\sum_{n=0}^{\infty}\frac{(1-h)_n(2h)_n(3n+3h+1)}{(-8)^n(h/2+1)_n(h/2+3/2)_n} \quad (4.3)$$

To derive (1.2) let h=1/3 in equation (4.3) and simplify:

$$\frac{3^{1/2}2^{2/3}\left[\Gamma(1/3)\right]^3}{9\pi}=\sum_{n=0}^{\infty}\frac{(2/3)_n(2/3)_n(3n+2)}{(-8)^n(7/6)_n(5/3)_n}$$

But $(2/3)_n(3n+2)=2(5/3)_n$ so we can write:

$$\frac{3^{1/2}2^{2/3}\left[\Gamma(1/3)\right]^3}{18\pi}=\sum_{n=0}^{\infty}\frac{(2/3)_n}{(-8)^n(7/6)_n}=2F1\left\{1,\frac{2}{3};\frac{7}{6};\frac{-1}{8}\right\} \quad (4.4)$$

(4.4) happens to also be a quadratically transformed hypergeometric function, which via (2.6) can be converted to an integral:

$$\frac{3^{1/2}2^{2/3}\left[\Gamma\left(1/3\right)\right]^3}{18\pi}=\frac{\Gamma\left(7/6\right)}{\Gamma\left(1/2\right)\cdot\Gamma\left(2/3\right)}\int_0^1\frac{x^{-1/3}\left(1-x\right)^{-1/2}}{1+x/8}dx$$

After some labor we obtain the simplification:

$$\frac{4\cdot 3^{1/2}\cdot\pi}{9}=\int_0^1\frac{x^{-1/3}\left(1-x\right)^{-1/2}}{1+x/8}dx \tag{4.5}$$

Letting

$$\begin{aligned}&k=1,\ s=1,\ a=-1/3,\ b=-1/2,\ P\left(x\right)=\left(1+x/8\right),\\&Q\left(x\right)=\left(8x-72\right),\ z=-72,\ a_1=8,\ a_0=-72\end{aligned}$$

We arrive at the identity:

$$\int_0^1\frac{x^{-1/3}\left(1-x\right)^{-1/2}}{1+x/8}dx=\int_0^1\frac{x^{-1/3}\left(1-x\right)^{-1/2}\left(8x-72\right)}{-72-x\left(1-x\right)}dx$$

Plugging the above values into (1.4) and simplifying through (2.5) we get (1.2):

$$\frac{7\cdot 3^{1/2}\left(\Gamma\left(1/3\right)\right)^3}{\pi\cdot 2^{1/3}}=\sum_{n=0}^{\infty}\frac{\left(\frac{2}{3}\right)_n\left(\frac{1}{2}\right)_n\left(102n+59\right)}{\left(\frac{13}{12}\right)_n\left(\frac{19}{12}\right)_n\left(-288\right)^n}$$

An even faster formula for $\Gamma\left(1/3\right)$ can be found that gives 3.6 digits/term though the following integral identity where we let s=2 and k=1:

$$\frac{1}{8}\int_0^1\frac{x^{-1/3}\left(1-x\right)^{-1/2}}{1+x/8}dx=\int_0^1\frac{x^{-1/3}\left(1-x\right)^{-1/2}\left(-x^2+10x-81\right)}{-648-x\left(1-x\right)^2}dx=\frac{\pi\sqrt{3}}{18}$$

Applying (1.4) and (2.5) and after some labor obtain:

$$\frac{91\cdot 3^{3/2}\left[\Gamma\left(1/3\right)\right]^3}{2^{1/3}\pi}=\sum_{n=0}^{\infty}\frac{\left(2/3\right)_n\left(1/4\right)_n\left(3/4\right)_n\left(8400n^2+9248n+2297\right)}{\left(19/18\right)_n\left(25/18\right)_n\left(31/18\right)_n\left(-4374\right)^n} \tag{4.6}$$

Higher Order Constants & the Limitations of Accelerating the Generalized Hypergeometric Function

There series we've been deriving so far involve constants that can be expressed though the hypergeometric function of the form $2F1\{1,x;y;z\}$. Such constants involve logarithms, root extractions, the gamma function, and inverse trigonometric functions. Higher order constants such as Catalans's Constant, Zeta 3, etc can't be expressed through $2F1\{1,x;y;z\}$, but require a more generalized hypergeometric function of the form: ${}_{q+1}F_q\left\{1,x_1,...,x_q;y_1;...;y_q;z\right\}$.

Unfortunately, the algorithm (1.4) isn't as efficient at generating rapidly converging series for constants that are only expressible though a hypergeometric function where q>1. To the best of my knowledge for (1.4) to work on higher order constants such as Zeta 3, the value of 's' must be set to zero. Accelerating the convergence of a q>1 hypergeometric function via (1.4) by setting s=0 equates to taking multiple terms of ${}_{q+1}F_q\left\{1,x_1,...,x_q;y_1;...;y_q;z\right\}$ 'k' terms at a time. So if k=2, the series generated by (1.4) would converge at a rate of z^2.

In this next example we will show how to accelerate the convergence of a hypergeometric function of the form $3F2\{1,x_1,x_2;y_1;y_2;z\}$ using (1.4), and then apply it to derive faster converging infinite series for Catalan's Constant and $\Gamma(1/4)$.

Begin with the integrals:

$$3F2\{1,x_1,x_2;y_1;y_2;z\}=\frac{\Gamma(y_2)}{\Gamma(x_2)\Gamma(y_2-x_2)}\int_0^1 x^{x_2-1}(1-x)^{y_2-x_2-1}2F1\{1,x_1;y_1;xz\}\,dx \tag{5.1}$$

$$2F1\{1,x_1;y_1;xz\}=\frac{\Gamma(y_1)}{\Gamma(x_1)\Gamma(y_1-x_1)}\int_0^1\frac{y^{x_1-1}(1-y)^{y_1-x_1-1}}{1-yxz}\,dy \tag{5.2}$$

Using the integral (5.2), choosing the values below for k and s, and performing polynomial long division we plug the following into (1.4):

$$s=0,\ k=2,\ b=(y_1-x_1-1),\ a=(x_1-1),\ z=\frac{1}{x^2z^2}$$
$$P(x,y,z)=(1-xyz),\ Q(x,y,z)=\left(\frac{y}{xz}+\frac{1}{x^2z^2}\right)$$

And obtain:

$$\int_0^1 \frac{y^{x_1-1}(1-y)^{y_1-x_1-1}}{P(x,y,z)}dy = \int_0^1 \frac{y^{x_1-1}(1-y)^{y_1-x_1-1}Q(x,y,z)}{\frac{1}{x^2z^2}-y^2}dy =$$

$$\frac{x^2z^2\cdot\Gamma(x_1)\Gamma(y_1-x_1)}{\Gamma(y_1)}\sum_{n=0}^{\infty}\frac{(x_1)_{2n}(xz)^{2n}}{(y_1)_{2n}}\left(\frac{x_1+2n}{xz(y_1+2n)}+\frac{1}{x^2z^2}\right) \tag{5.3}$$

Plugging (5.3) into (5.2), and then plugging that result into (5.1) and simplifying we get:

$$\frac{\Gamma(y_2)}{\Gamma(x_2)\Gamma(y_2-x_2)}\int_0^1 x^{x_2-1}(1-x)^{y_2-x_2-1}\sum_{n=0}^{\infty}\frac{(x_1)_{2n}(xz)^{2n}}{(y_1)_{2n}}\left(\frac{xz(x_1+2n)}{y_1+2n}+1\right)dx$$

$$=3F2\{1,x_1,x_2;y_1;y_2;z\} \tag{5.4}$$

Then we extract and convert the following integrals from (5.4) into pochhammer & gamma notation via the beta function:

$$\int_0^1 x^{2n+x_2-1}(1-x)^{y_2-x_2-1}dx = \frac{\Gamma(2n+x_2)\Gamma(y_2-x_2)}{\Gamma(2n+y_2)} = \frac{(x_2)_{2n}\Gamma(x_2)\Gamma(y_2-x_2)}{(y_2)_{2n}\Gamma(y_2)}$$

$$\int_0^1 x^{2n+x_2}(1-x)^{y_2-x_2-1}dx = \frac{\Gamma(2n+x_2+1)\Gamma(y_2-x_2)}{\Gamma(2n+y_2+1)} = \frac{(x_2+2n)(x_2)_{2n}\Gamma(x_2)\Gamma(y_2-x_2)}{(y_2+2n)(y_2+1)_{2n}\Gamma(y_2)} \tag{5.5}$$

Then plugging (5.5) back into (5.4) and simplifying gives the final result:

$$3F2\{1,x_1,x_2;y_1;y_2;z\} = \sum_{n=0}^{\infty}\frac{(x_1)_{2n}(x_2)_{2n}(z)^{2n}}{(y_1)_{2n}(y_2)_{2n}}\left(\frac{z(x_1+2n)(x_2+2n)}{(y_1+2n)(y_2+2n)}+1\right) \tag{5.6}$$

The procedure used to derive (5.6) can be generalized:

$$_{q+1}F_q\{1,x_1,...,x_q;y_1;...;y_q;z\} = \sum_{n=1}^{\infty}\left(\left(\left(z\prod_{g=1}^{q}\frac{x_g+2n}{y_g+2n}\right)+1\right)\prod_{g=1}^{q}\frac{(x_g)_{2n}}{(y_g)_{2n}}\right) \tag{5.7}$$

With regards to Catalan's Constant it is well-known that

$$3F2\left\{1,1,\frac{1}{2};\frac{3}{2};\frac{3}{2};\frac{1}{4}\right\}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\binom{2n}{n}}=\frac{\pi}{3}\ln\left(2-\sqrt{3}\right)+\frac{8}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \qquad (5.8)$$

Plugging the hyqpergeometric terms of (5.8) into (5.6) and simplifying gives an infinite series for (5.8) that converges twice as fast:

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\binom{2n}{n}}=\sum_{n=0}^{\infty}\frac{40n^2+54n+19}{\left((4n+1)(4n+3)\right)^2\binom{4n}{2n}} \qquad (5.9)$$

Using a transformation on (5.2) letting k=3 gives:

$$4\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\binom{2n}{n}}=\sum_{n=0}^{\infty}\frac{6804n^4+17172n^3+15903n^2+6405n+956}{\left((6n+1)(6n+3)(6n+5)\right)^2\binom{6n}{3n}} \qquad (5.10)$$

In addition, rapidly converging formulas for $\Gamma(1/4)$ can be derived using (5.6) based on accelerating a quadratic transformation. Begin with the following quadratically transformed hypergeometric functions:

$$2F1\left\{\frac{1}{4},\frac{1}{4};1;\frac{-1}{8}\right\}=\frac{\sqrt{\pi}}{2^{1/4}\left[\Gamma(3/4)\right]^2} \qquad (5.11)$$

$$2F1\left\{\frac{1}{4},\frac{3}{4};1;\frac{1}{9}\right\}=\frac{\sqrt{3\pi}\left[\Gamma(1/4)\right]^2}{4\pi^2} \qquad (5.12)$$

(Note: Unlike equation (4.4), (5.11) and (5.12) can’t be derived with (1.4). Proofs regarding quadratic hypergeometric transformations is given in [7].)

However, (5.11) is equal to $3F2\left\{1,\frac{1}{4},\frac{1}{4};1;1;\frac{-1}{8}\right\}$ and (5.12) can be written as $3F2\left\{1,\frac{1}{4},\frac{3}{4};1;1;\frac{1}{9}\right\}$. Plugging these values into (5.6) and simplifying via (2.5) gives

$$\frac{36\sqrt{3}\left[\Gamma(1/4)\right]^2}{\pi^{3/2}}=\sum_{n=0}^{\infty}\frac{(640n^2+608n+147)(8n)!}{\left[(2n+1)!\right]^2(4n)!24^{4n}} \qquad (5.13)$$

$$\frac{128\pi^{1/2}}{2^{1/4}\left[\Gamma(3/4)\right]^2}=\sum_{n=0}^{\infty}\left(448n^2+448n+127\right)\left(\frac{(1/8)_n(5/8)_n}{2^n(2n+1)!}\right)^2 \tag{5.14}$$

The formulas (5.13) and (5.14) conclude the paper. So far, we've proven and showed examples how implementing algorithm (1.4) involving polynomial division can be used to derive elegant, rapidly converging infinite series for various mathematical constants, as well as accelerate the convergence of existing series as in examples (5.13, 5.14, 5.9, and 5.10). To the best of my knowledge, most of the series derived in this paper are original, although the procedure involving polynomial division isn't. However, I have yet to see an explicit, closed form algorithms like (1.3) or (1.4) for deriving a wide spectrum of infinite series.

An open problem related to this paper is if there exists a transformation using (1.4) or a variant of it for ${}_{q+1}F_q\left\{1,x_1,...,x_q;y_1;...;y_q;z\right\}$ q>1 where s >0. My best attempts to derive closed form, transformation for non trivial 3F2 or higher hypergeometric functions for s>0 have failed. By non trivial what I mean is that $x_n - y_n$ cannot be a positive integer. $3F2\{1,4,5/2;3;1/3;z\}$, for example, is trivial because 4-3 is an integer. $3F2\{1,3,5/2;4;1/3;z\}$, on the other hand, isn't trivial because 3-4 is a negative integer. Also, 5/2-1/3 isn't an integer.

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